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An extremal problem for polynomials with a prescribed multiple zero

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Abstract

We obtain a sharp inequality for trigonometric polynomials which have a double zero at a specified point. A similar extremal problem has been studied by R.P. Boas Jr.

Let t be a trigonometric polynomial and let some *incomplete* information about t, like upper bounds of its modulus or the modulus of its derivatives on a point set Λ, be given. We wish to find how large its modulus can be at a point (or at a point set) outside of Λ. The global treatment of this problem can be seen in the book of Rahman and Schmeisser [2, Chapter 12], where the detailed discussion of the following theorem due to Boas Jr. [1, Theorem 2, p. 43] is given:

Theorem A. *Let S_n be a trigonometric polynomial of degree n with $S_n(0) = 0$ and $|S_n((2k+1)\pi/2n)| \leqslant 1$ $(k = 0, 1, \ldots, 2n-1)$; then*

$$\bigl|S_n(\theta)\bigr| \leqslant |\sin n\theta| \quad \bigl(|\theta| \leqslant \pi/2n\bigr). \tag{1}$$

There is equality in (1) *for some θ (and hence for all θ) if and only if $S_n(\theta)$ is a constant multiple of $\sin n\theta$. If S_n is real, the hypothesis $S_n(0) = 0$ can be replaced by $S_n(0) \leqslant 0$. If $\pi \geqslant \theta_0 > \pi/2n$, there is a trigonometric polynomial S of degree n with $|S(\theta)| \leqslant 1$, $S(0) = 0$ and $S(\theta_0) = 1$.*

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Before formulating our new result, we shall construct a trigonometric polynomial that plays a crucial role in the investigation of this paper. Let us consider the Chebyshev polynomial of the first kind of degree n, $T_n(x) := \cos n \arccos x$, and the linear transformation $y(x) := x\cos^2 \pi/4n - \sin^2 \pi/4n$ that maps the interval $[-1, 1]$ onto the interval $[-1, \cos \pi/2n]$. Then, the even trigonometric polynomial

$$\Psi(\theta) := -T_n\bigl(y(\cos\theta)\bigr) = -T_n\left(\cos^2\frac{\pi}{4n}\,\cos\theta - \sin^2\frac{\pi}{4n}\right)$$

is of degree n and has the following properties:

(1) The polynomial $\Psi(\theta)$ has a double zero at $\eta_0 = 0$, and $2n-2$ additional, simple and real zeros $0 < \eta_1 < \eta_2 < \cdots < \eta_{2n-2} < 2\pi$. More precisely,

$$\eta_\nu = 2\arccos\left(\left(\cos\frac{\pi}{4n}\right)^{-1}\cos\frac{(2\nu+1)\pi}{4n}\right), \quad \nu = 1, \ldots, n-1,$$

and $\eta_\nu := 2\pi - \eta_{2n-1-\nu}$, $\nu = n, \ldots, 2n-2$.

(2) The polynomial $\Psi(\theta)$ has $2n-1$ points of local extrema in the interval $[0, 2\pi)$:

$$\theta_\nu = 2\arccos\left(\left(\cos\frac{\pi}{4n}\right)^{-1}\cos\frac{\nu\pi}{2n}\right), \quad \nu = 1, \ldots, 2n-1.$$

Moreover, $\theta_n = \pi$ and $\Psi(\theta_\nu) = (-1)^{\nu+1}$.

Now we are ready to formulate our new result.

Theorem 1. *Let t be a trigonometric polynomial of degree at most n such that $t(0) = t'(0) = 0$. Furthermore, let us assume that*

$$\bigl|t(\theta_\nu)\bigr| \leqslant 1 \quad \textit{for } \nu = 1, \ldots, 2n-1.$$

Then

$$|t(\theta)| < \Psi(\theta) \quad \textit{for } 0 < |\theta| < \theta_1 \tag{2}$$

unless $t(\theta) = e^{i\gamma}\Psi(\theta)$ for some real γ.

Proof. We divide the proof into two parts. In the part A of the proof we shall show that $|t(\theta)| \leqslant \Psi(\theta)$ for $-\theta_1 \leqslant \theta \leqslant \theta_1$. In the part B, all possible cases of equality $|t(\theta^*)| = \Psi(\theta^*)$ at some point $\theta^* \in \{\theta\colon 0 < |\theta| < \theta_1\}$ shall be considered.

A. Let us suppose to the contrary that there is a point $\theta^* \in (0, \theta_1)$ such that $|t(\theta^*)| > \Psi(\theta^*)$ ($\Psi(\theta^*) > 0$). In view of the representation $t(\theta^*) = |t(\theta^*)|e^{i\varphi_*}$, the trigonometric polynomial

$$s(\theta) := \Re\bigl\{e^{-i\varphi_*}t(\theta)\bigr\}$$

has the following properties:

(a) $s(\theta)$ is of degree n and satisfies $s(0) = s'(0) = 0$;
(b) $|s(\theta_\nu)| \leqslant |\Psi(\theta_\nu)| = 1$, $\nu = 1, \ldots, 2n-1$;
(c) $s(\theta^*) = |t(\theta^*)| > \Psi(\theta^*)$.

Let $\lambda \in (0,1)$ be such that $\lambda s(\theta^*) > \Psi(\theta^*)$; $|\lambda s(\theta_\nu)| < 1$ (strictly less than 1) for $\nu = 1, \ldots, 2n-1$ and $\lambda s''(0) \neq \Psi''(0)$. Note that $\Psi''(0) \neq 0$. In view of the properties of $\lambda s(\theta)$ and $\Psi(\theta)$ the trigonometric polynomial $\Psi(\theta) - \lambda s(\theta)$ must have at least one zero in each of the intervals $(\theta_\nu, \theta_{\nu+1})$ for $\nu = 1, \ldots, 2n-2$ and at least 3 zeros in $[0, \theta_1)$. Note that $\theta_{2n-1} = 2\pi - \theta_1$. Summing up, the trigonometric polynomial $\Psi(\theta) - \lambda s(\theta)$ has at least $2n+1$ zeros in $[0, 2\pi)$ and so must be identically zero. However, $\Psi(\theta^*) - \lambda s(\theta^*) \neq 0$ and we are led to a contradiction. We conclude that

$$\left|t(\theta)\right| \leqslant \Psi(\theta) \quad \text{for } 0 \leqslant |\theta| \leqslant \theta_1. \tag{3}$$

B. Here we discuss all possible cases of equality in (3). Suppose that there is a point $\theta^* \in (-\theta_1, \theta_1)\backslash\{0\}$ such that $|t(\theta^*)| = \Psi(\theta^*)$. Without any restriction we can assume $\theta^* \in (0, \theta_1)$. Let $t(\theta^*) = |t(\theta^*)|e^{i\varphi_*}$. Then, the real trigonometric polynomial

$$s(\theta) := \Re\left\{e^{-i\varphi_*}\, t(\theta)\right\}$$

is of degree at most n and possesses the following properties:

(a) $s(0) = s'(0) = 0$;
(b) $|s(\theta_\nu)| \leqslant |\Psi(\theta_\nu)| = 1$, $\nu = 1, \ldots, 2n-1$;
(c) $s(\theta^*) = |t(\theta^*)| = \Psi(\theta^*) > 0$.

In view of the part A of the proof we have $s(\theta) \leqslant \Psi(\theta)$ for $\theta \in [0, \theta_1]$. Hence, θ^* is a zero of (even) multiplicity at least two for the trigonometric polynomial $\Psi(\theta) - s(\theta)$. Let us denote by $\mathbf{Z}[0, \theta_{2\nu-1}]$ the number of zeros in the interval $[0, \theta_{2\nu-1}]$.

We claim that $\mathbf{Z}[0, \theta_{2\nu-1}] \geqslant 2\nu + 1$ if $s(\theta_{2\nu-1}) < 1$ or $s(\theta_{2\nu-1}) = 1$ and $s'(\theta_{2\nu-1}) < 0$; whereas $\mathbf{Z}[0, \theta_{2\nu-1}] \geqslant 2\nu + 2$ if $s(\theta_{2\nu-1}) = 1$ and $s'(\theta_{2\nu-1}) \geqslant 0$.

For $\nu = 1$ this is obvious. Now, let us assume that for $\nu \leqslant k$ the above claim holds. We shall prove it for $\nu = k+1$ by making use of the induction supposition. We consider two cases:

I. Suppose that either $s(\theta_{2k-1}) < 1$ or $s(\theta_{2k-1}) = 1$ and $s'(\theta_{2k-1}) < 0$. Then, in the case $s(\theta_{2k-1}) = 1$ and $s'(\theta_{2k-1}) < 0$ we can choose $\delta > 0$ and sufficiently small such that $s(\theta_{2k-1} + \delta) < \Psi(\theta_{2k-1} + \delta)$. So, without any restriction, in both cases we can suppose $s(\theta_{2k-1}) < 1$.

(a) Let $s(\theta_{2k+1}) < 1$ or $s(\theta_{2k+1}) = 1$ and $s'(\theta_{2k+1}) < 0$. Analogously, without any restriction, we can only assume $s(\theta_{2k+1}) < 1$. By induction we have at least $2k+1$ zeros for $\Psi - s$ in the interval $[0, \theta_{2k-1}]$. On the other hand, $\Psi - s$ will have at least 2 zeros in $(\theta_{2k-1}, \theta_{2k+1}]$. Hence, the zeros of $\Psi - s$ in $[0, \theta_{2k+1}]$ will be at least $2k + 1 + 2 = 2(k+1) + 1$.

(b) Let $s(\theta_{2k+1}) = 1$ and $s'(\theta_{2k+1}) \geqslant 0$. Then, $\Psi - s$ has at least three zeros in $(\theta_{2k-1}, \theta_{2k+1}]$, so at least $2(k+1) + 2$ zeros in $[0, \theta_{2k+1}]$.

II. Here we suppose $s(\theta_{2k-1}) = 1$ and $s'(\theta_{2k-1}) \geqslant 0$. By induction, $\Psi - s$ has at least $2k+2$ zeros in $[0, \theta_{2k-1}]$.

(a) Let either $s(\theta_{2k+1}) < 1$ or $s(\theta_{2k+1}) = 1$ and $s'(\theta_{2k+1}) < 0$. Then, $\Psi - s$ will have at least one zero in $(\theta_{2k-1}, \theta_{2k+1}]$ and $\Psi - s$ will have at least $2(k+1) + 1$ zeros.

(b) Suppose $s(\theta_{2k+1}) = 1$ and $s'(\theta_{2k+1}) \geqslant 0$. Then, $\Psi - s$ will have at least 2 zeros in $(\theta_{2k-1}, \theta_{2k+1}]$. Hence, $\Psi - s$ will have at least $2(k+1) + 2$ zeros in $[0, \theta_{2k+1}]$.

In view of the above considerations, we obtain that the trigonometric polynomial $\Psi - s$ of degree at most n must have at least $2n+1$ zeros in the interval $[0, \theta_{2n-1}] \subset [0, 2\pi)$ and it should be identically zero which is $s(\theta) \equiv \Psi(\theta)$. From the representation

$$s(\theta) = \Re\left\{e^{-i\varphi^*} t(\theta)\right\}$$

it follows that the trigonometric polynomial $\Im\{e^{-i\varphi^*} t(\theta)\}$ has the following properties:

(i) $\Im\{e^{-i\varphi^*} t(\theta_\nu)\} = 0$, $\nu = 1, \ldots, 2n-1$;
(ii) it has a zero at $\eta_0 = 0$ with multiplicity at least 2.

From (i) and (ii) it follows that $\Im\{e^{-i\varphi^*} t(\theta)\} \equiv 0$ and

$$t(\theta)e^{-i\varphi^*} = \Re\left\{t(\theta)e^{-i\varphi^*}\right\} = s(\theta) = \Psi(\theta).$$

We conclude that equality at some point $\theta^* \in (-\theta_1, 0) \cup (0, \theta_1)$ is possible in (3) only when $t(\theta) = e^{i\gamma}\Psi(\theta)$ for some $\gamma \in \mathbb{R}$. $\square$

The next corollary gives an upper bound for the second derivative at the point zero of a trigonometric polynomial under the conditions of Theorem 1.

Corollary 1. *Let t be a trigonometric polynomial of degree at most n such that $t(0) = t'(0) = 0$. Furthermore, let us suppose that*

$$\left|t(\theta_\nu)\right| \leqslant 1 \quad \text{for } \nu = 1, \ldots, 2n-1.$$

Then, we have

$$\left|t''(0)\right| \leqslant \Psi''(0) = \frac{n}{2}\cot\frac{\pi}{4n}, \tag{4}$$

where equality holds if and only if $t(\theta) = e^{i\gamma}\Psi(\theta)$ for some real γ.

Proof. Let $t(\theta)$ be a trigonometric polynomial satisfying the conditions of Corollary 1. Then,

$$\left|\frac{t(\theta)}{\theta^2}\right| \leqslant \frac{\Psi(\theta)}{\theta^2} \quad \text{for } \theta \in [-\theta_1, \theta_1].$$

After taking the limit in the above inequality when $\theta \to 0$, $\theta \neq 0$, we obtain $|t''(0)| \leqslant \Psi''(0)$ by using L'Hopital's rule. Now, suppose that for a trigonometric polynomial t we have an equality in (4). Then, the real trigonometric polynomial $s(\theta) := \Re\{e^{-i\phi}t(\theta)\}$ satisfies the conditions of Theorem 1. Hence, $s(\theta) \leqslant \Psi(\theta)$ for $\theta \in [-\theta_1, \theta_1]$. The representation $t''(0) = e^{i\phi}|t''(0)|$ shows that $s''(0) = \Psi''(0)$. Assuming $s'''(0) \neq \Psi'''(0)$ we are led to a contradiction by making use of Taylor's formula up to the third derivative. From here, $s'''(0) = \Psi'''(0)$, which means that the point 0 is a zero of $\Psi - s$ with multiplicity at least 4. Now, proceeding as in the proof of Theorem 1, we conclude that the case of equality in (4) is possible only when $t(\theta) \equiv e^{i\gamma}\Psi(\theta)$ for some real γ. This completes the proof. $\square$

Example. With the following example we show that the point set

$$\{\theta\colon\ \theta \in (0,\theta_1) \cup (\theta_{2n-1}, 2\pi)\}$$

is *the maximal point set* in

$$(0,2\pi)\backslash\{\theta_1,\ldots,\theta_{2n-1}\} \quad (n \geqslant 2),$$

where the inequality $|t(\theta)| \leqslant \Psi(\theta)$ holds for any trigonometric polynomial t, satisfying the conditions of Theorem 1. Indeed, let θ_* be an arbitrarily chosen point from the set $(0,2\pi)\backslash\{(0,\theta_1) \cup (\theta_{2n-1}, 2\pi)\}$ and different from θ_j, $j = 1,\ldots,2n-1$. Then, $\theta_* \in (\theta_{\nu_0}, \theta_{\nu_0+1})$ for some $\nu_0 \in \{1,\ldots,2n-2\}$.

We consider the unique trigonometric polynomial t_{ν_0} of degree at most n satisfying the $2n+1$ Hermite type interpolation conditions:

$$t_{\nu_0}(0) = 0, \qquad t'_{\nu_0}(0) = 0, \qquad t_{\nu_0}(\theta_v) = (-1)^{\nu-\nu_0}, \qquad \nu = 1,\ldots,\nu_0,$$

and

$$t_{\nu_0}(\theta_v) = (-1)^{\nu-\nu_0-1}, \quad \nu = \nu_0+1,\ldots,2n-1.$$

Let us suppose that the value of t_{ν_0} at the point θ_* is less than or equal to 1, i.e., $t_{\nu_0}(\theta_*) \leqslant 1$. Then, the zeros of the first derivative of t_{ν_0} in $[0,2\pi)$ will be at least $2n+1$ which means that t_{ν_0} must be a constant. Obviously, this is impossible. Hence, $t_{\nu_0}(\theta_*) > 1 > \Psi(\theta_*)$. We conclude that Ψ is not an upper bound for our polynomial class at the point θ_*.

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